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CODIMENSION ONE ISOMETRIC IMMERSIONS BETWEEN LORENTZ SPACES

BY

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In memory of my father, Lucius Kingman Graves

ABSTRACT. The theorem of Hartman and Nirenberg classifies codimension one isometric immersions between Euclidean spaces as cylinders over plane curves. Corresponding results are given here for Lorentz spaces, which are Euclidean spaces with one negative-definite direction (also known as Minkowski spaces). The pivotal result involves the completeness of the relative nullity foliation of such an immersion. When this foliation carries a nondegenerate metric, results analogous to the Hartman-Nirenberg theorem obtain. Otherwise, a new description, based on particular surfaces in the three-dimensional Lorentz space, is required.

The theorem of Hartman and Nirenberg [HN] says that, up to a proper motion of $\mathbf{E}^{n+1}$, all isometric immersions of $\mathbf{E}^n$ into $\mathbf{E}^{n+1}$ have the form

$$\mathrm{id} \times c: \mathbf{E}^{n-1} \times \mathbf{E}^1 \to \mathbf{E}^{n-1} \times \mathbf{E}^2 \tag{0.1}$$

where $c: \mathbf{E}^1 \to \mathbf{E}^2$ is a unit speed plane curve and the factors in the product are orthogonal. A proof of the Hartman-Nirenberg result also appears in [N_1]. The major step in proving the theorem is showing that the relative nullity foliation, which is spanned by those tangent directions in which a local unit normal field is parallel, has complete leaves. These leaves yield the $\mathbf{E}^{n-1}$ factors in (0.1). The one-dimensional complement of the relative nullity foliation gives rise to the curve c.

This paper studies isometric immersions of $\mathbf{L}^n$ into $\mathbf{L}^{n+1}$, where $\mathbf{L}^n$ denotes the n-dimensional Lorentz (or Minkowski) space. Its chief goal is the classification, up to a proper motion of $\mathbf{L}^{n+1}$, of all such immersions.

Part I introduces the terminology and states the elementary results required to achieve the desired classification. In particular, null curves–curves in $\mathbf{L}^n$ whose tangent vectors, although nonzero, have zero "length"–and appropriate frames for them are discussed, the theory of Lorentz hypersurfaces in $\mathbf{L}^{n+1}$ is described, and examples are exhibited.

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The major task of demonstrating the completeness of the leaves of the relative nullity foliation is accomplished in Part II. The case where the relative nullity spaces inherit nondegenerate metrics can be handled virtually in the same manner as the Euclidean case. However, when the relative nullities are "degenerate" (defined in §6), different techniques are required. It is shown in passing that the cases are mutually exclusive in nontrivial cases.

Part III contains the classification results. When the relative nullity spaces are nondegenerate, the immersion has one of the following forms:

$$\mathrm{id} \times c\colon \mathbf{E}^{n-1} \times \mathbf{L}^1 \to \mathbf{E}^{n-1} \times \mathbf{L}^2, \tag{0.2}$$

$$\mathrm{id} \times c\colon \mathbf{L}^{n-1} \times \mathbf{E}^1 \to \mathbf{L}^{n-1} \times \mathbf{E}^2. \tag{0.3}$$

In (0.2) and (0.3), c is unit-speed in the sense that $\langle dc/ds, dc/ds\rangle$ is -1 and $+1$, respectively. Two theorems cover the case when the relative nullities are degenerate. Theorem (9.7) characterizes the class of isometric immersions of $\mathbf{L}^2$ into $\mathbf{L}^3$ with this property as the class of B-scroll immersions, which are described by Example (3.9) and Theorem (3.13). Then the general isometric immersion $\mathbf{L}^n \to \mathbf{L}^{n+1}$ with degenerate relative nullity, according to Theorem (9.8), has the form

$$\mathrm{id} \times g\colon \mathbf{E}^{n-2} \times \mathbf{L}^2 \to \mathbf{E}^{n-2} \times \mathbf{L}^3 \tag{0.4}$$

where g is a B-scroll immersion.

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Part I. Elementary Lorentz geometry

1. Remarks on $\mathbf{L}^n$ and the Lorentz group. Consider the real n-dimensional vector space $\mathbf{R}^n$ with the standard basis $\{e_0, e_1, \ldots, e_{n-1}\}$. Let $\langle\ ,\ \rangle$ denote the (indefinite) inner product on $\mathbf{R}^n$ whose matrix with respect to the standard basis is

$$S = \left[\begin{array}{c|ccc} -1 & & & \\ \hline & 1 & & \\ & & \ddots & \\ & & & 1 \end{array}\right].$$

Then $\langle\ ,\ \rangle$ is called the *Lorentz metric* on $\mathbf{R}^n$; $\mathbf{R}^n$ together with this metric is called the *n-dimensional Lorentz space* and is denoted by $\mathbf{L}^n$. A vector X in $\mathbf{L}^n$ is called *time-like* or *space-like* according to whether $\langle X, X\rangle$ is negative or positive. If $|\langle X, X\rangle| = 1$, then X is called a unit vector of its type. If

$\langle X, X \rangle = 0$, then X is called a light-like or null vector. Time- and light-like vectors are *causal* vectors.

The proof of the following result is standard.

THEOREM (1.1). *Let P be a k-dimensional subspace of $\mathbf{L}^n$. Then exactly one of the following statements about P is true*:

(i) $P = \mathbf{L}^k$, *and* $\langle\, , \rangle|p$ *is nondegenerate*;

(ii) $P = \mathbf{E}^k$, *and* $\langle\, , \rangle|p$ *is nondegenerate*;

(iii) $\langle\, , \rangle|p$ *is degenerate, and in this case* (*and only in this case*)

$$P = \mathbf{E}^{k-1} \oplus \mathrm{Span}\{\xi\}$$

where $\langle \xi, \xi \rangle = 0$ *and* ξ *is orthogonal to* $\mathbf{E}^{k-1}$.

Define $\mathrm{O}(1, n-1)$ to be the subgroup of $\mathrm{GL}(n, \mathbf{R})$ consisting of those transformations which preserve the Lorentz metric. In terms of matrices, we may write

$$\mathrm{O}(1, n-1) = \{U \in \mathrm{GL}(n, \mathbf{R}) \colon USU^T = S\} \tag{1.2}$$

where U^T denotes the transpose of U. $\mathrm{O}(1, n-1)$ is the group of *Lorentz transformations* on $\mathbf{R}^n$. An ordered basis for $\mathbf{L}^n$ is a *frame*; $(e_0, \ldots, e_{n-1})$ is the *standard* (*orthonormal*) *frame*. An *orthonormal frame* for $\mathbf{L}^n$ is the image of the standard frame under a Lorentz transformation.

The subgroup $\mathrm{SO}^+(1, n-1)$ of $\mathrm{O}(1, n-1)$ whose elements U satisfy $\det U = 1$ and $\langle Ue_0, e_0 \rangle \leqslant -1$ is the component of the identity in $\mathrm{O}(1, n-1)$ and is called the group of *proper Lorentz transformations*.

Define $o(1, n-1)$ to be the set of $n \times n$ matrices Y satisfying $SY^TS = -Y$. It is the Lie algebra of $\mathrm{SO}^+(1, n-1)$, and each element of it has the form

$$\left[\begin{array}{c|c} 0 & X^T \\ \hline X & K \end{array}\right] \tag{1.3}$$

where $X \in \mathbf{R}^{n-1}$ and K is an $(n-1) \times (n-1)$ skew-symmetric matrix.

LEMMA (1.4). *Let* $s \mapsto U(s)$ *be a curve in* $\mathrm{O}(1, n-1)$. *Then*

$$\frac{dU}{ds} SU^T + US \frac{dU^T}{ds} = 0.$$

PROPOSITION (1.5). *Let* $s \mapsto U(s)$ *be a curve in* $\mathrm{GL}(n, \mathbf{R})$. *Then* $U(s)$ *lies in* $\mathrm{O}(1, n-1)$ *if and only if* $U(s_0) \in \mathrm{O}(1, n-1)$ *for some* $s_0 \in \mathbf{R}$ *and* $U^{-1}dU/ds \in o(1, n-1)$.

2. Null curves and frames in $\mathbf{L}^3$. In this section, the ambient space for all curves, frames, etc., is $\mathbf{L}^3$.

A *null frame* is an ordered triple of vectors (or, from another point of view, a matrix):

$$F = (A, B, C) = \begin{bmatrix} a_0 & b_0 & c_0 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix}$$

where A and B are null vectors satisfying $\langle A, B \rangle = -1$, C is a unit space-like vector orthogonal to the Lorentz plane spanned by A and B, and $\det F = \pm 1$. To any null frame there is a uniquely determined *associated orthonormal frame* defined by

$$L(F) = \left(\frac{1}{\sqrt{2}}(A + B), \frac{1}{\sqrt{2}}(A - B), C \right).$$

A null frame F will be called *proper* if $L(F) \in SO^+(1, 2)$. One refers to

$$N = \frac{1}{\sqrt{2}} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

as the "standard null frame": the associated orthonormal frame is the standard orthonormal frame (e_0, e_1, e_2). More generally,

LEMMA (2.1). *If F is a null frame, then $L(F) = F \cdot N^{-1}$.*

PROPOSITION (2.2). *The set of proper null frames is precisely the right coset of* $SO^+(1, 2)$ *in* $GL(3, \mathbf{R})$ *determined by N.*

PROPOSITION (2.3). *Let $s \mapsto F(s)$ be a curve in* $GL(3, \mathbf{R})$. *Then* $F(s) \in O(1, 2) \cdot N$ *if and only if* $F(s_0) \in (1, 2) \cdot N$ *for some* $s_0 \in \mathbf{R}$ *and* $F^{-1} dF/ds$ *has the form*

$$\begin{bmatrix} k_1 & 0 & k_3 \\ 0 & -k_1 & k_2 \\ k_2 & k_3 & 0 \end{bmatrix}.$$

REMARK. In Proposition (2.3), if, for some $s_0 \in \mathbf{R}$, $F(s_0)$ is proper, then $F(s)$ is proper for all s.

A curve $s \mapsto F(s)$ in $SO^+(1, 2) \cdot N$ is a (*proper*) *frame curve.* If $F(s) = (A(s), B(s), C(s))$ is a frame curve, then Proposition (2.3) implies that the following system of differential equations is satisfied:

$$\begin{aligned} dA/ds &= k_1(s)A(s) \qquad\qquad\qquad + k_2(s)C(s),\\ dB/ds &= \qquad\qquad\qquad -k_1(s)B(s) + k_3(s)C(s),\\ dC/ds &= k_3(s)A(s) + k_2(s)B(s). \end{aligned} \tag{2.4}$$

The equations which comprise (2.4) will be called the *Frenet equations* for F. It is clear that, given a proper null frame F and a system (2.4), there is a unique frame curve $F(s)$ which satisfies $F(0) = F$ and has (2.4) as its Frenet equations.

A *null curve* in $\mathbf{L}^3$ is a curve $s \mapsto x(s)$ all of whose tangent vectors are nonzero and null ($\langle dx/ds, dx/ds \rangle = 0$). Since this condition is unaffected by a change of parameter, there is no analog of the arc-length parameter which curves in Euclidean space possess. There is, however, the following proposition about null curves.

PROPOSITION (2.5). *Let $x(s)$ be a null curve in $\mathbf{L}^3$ such that $x(0) = 0$. Then $x(s)$ lies within the causal cone of $\mathbf{L}^3$.*

PROOF. If $x(s) = (x^0(s), x^1(s), x^2(s))$ is the curve, then it is assumed that $x^i(0) = 0$ ($i = 0, 1, 2$) and

$$\left(\frac{dx^0}{ds}\right)^2 = \left(\frac{dx^1}{ds}\right)^2 + \left(\frac{dx^2}{ds}\right)^2.$$

We may assume that $dx^0/ds > 0$. Let $\tau(s) = (x^1(s), x^2(s))$ be the projection of $x(s)$ onto the Euclidean plane generated by $\{e_1, e_2\}$. Then the arc length from $\tau(0)$ to $\tau(s)$ is

$$\int_0^s \|\tau'(u)\|\, du = \int_0^s \left(\left(\frac{dx^1}{du}\right)^2 + \left(\frac{dx^2}{du}\right)^2\right)^{1/2} du$$
$$= \int_0^s \frac{dx^0}{du}\, du = x^0(s).$$

This arc length must be at least the length of the line segment from $\tau(0)$ to $\tau(s)$, which is $\|\tau(s)\| = ((x^1(s))^2 + (x^2(s))^2)^{1/2}$. Hence,

$$0 \geqslant -(x^0(s))^2 + (x^1(s))^2 + (x^2(s))^2,$$

which says that $x(s)$ lies in the causal cone of $\mathbf{L}^3$. Q.E.D.

A *frame* for a null curve $x(s)$ is a proper frame curve $F(s) = (A(s), B(s), C(s))$ such that dx/ds is a positive scalar multiple of $A(s)$; $x(s)$ is said to be *framed* by $F(s)$. Frames for null curves are not unique: if $(A(s), B(s), C(s))$ frames a given null curve, so does $(\frac{1}{2}(A(s), 2B(s), C(s)))$. Therefore, the curve and a frame must be given together. If (x, F) is a framed null curve, then one has the Frenet equations (2.4) with $dx/ds = k_0(s)A(s)$, $k_0(s) > 0$. If a point y, a proper null frame F, and functions $k_i(s)$ ($i = 0, 1, 2, 3$) with $k_0(s) > 0$ are specified, there is a unique framed null curve $(x(s), F(s))$ which satisfies $x(0) = y$, $F(0) = F$, and whose Frenet equations are (2.4) with $F = (A, B, C)$ and $dx/ds = k_0(s)A(s)$, $k_0(s) > 0$.

EXAMPLE (2.6). FRAMED NULL CURVE. Using coordinates with respect to the standard basis of $\mathbf{L}^3$, define (x, F) as follows:

$$F(s): \begin{cases} x(s) = & \left(\frac{4}{3}s^3 + s,\ \frac{4}{3}s^3 - s,\ 2s^2\right), \\ A(s) = & (4s^2 + 1, 4s^2 - 1, 4s), \\ B(s) = & \left(\frac{1}{2}, \frac{1}{2}, 0\right), \\ C(s) = & (-2s, -2s, -1). \end{cases}$$

The Frenet equations for (x, F) are simple: let $k_0(s) = 1$, $k_2(s) = -4$, and $k_1(s) = k_3(s) = 0$.

In Example (2.6), $k_0(s) = 1$ and $k_1(s) \equiv 0$. Thus $A(s)$ is the tangent vector (rather than its direction) and $C(s)$ is analogous to the principal normal in the standard Frenet frame for a curve in $\mathbf{E}^3$. More generally, a *Cartan-framed curve* is a framed null curve $(x(s), F(s))$ whose Frenet equations have the form

$$\frac{dx}{ds} = A(s), \qquad \frac{dA}{ds} = k_2(s)C(s), \qquad \frac{dB}{ds} = k_3(s)C(s), \qquad \frac{dC}{ds} = k_3(s)A(s) + k_2(s)B(s). \tag{2.7}$$

Bonnor [B] has made a thorough study of null curves in $\mathbf{L}^4$, and his methods may be adapted to examine null curves in $\mathbf{L}^3$ (or in $\mathbf{L}^n$, for any $n \geqslant 2$).

3. Theory of Lorentz hypersurfaces in $\mathbf{L}^{n+1}$. A hypersurface M in $\mathbf{L}^{n+1}$ is a *Lorentz hypersurface* if the tangent space to M at each point of M inherits a Lorentz metric from $\mathbf{L}^{n+1}$. If $f\colon M \to \mathbf{L}^{n+1}$ is an isometric immersion, then $f(M)$ is a Lorentz hypersurface which may simply be denoted by M. The Lorentz metric on $M, f(M)$, or $\mathbf{L}^{n+1}$ will be denoted by $\langle\ ,\ \rangle$; the context will resolve any ambiguity. Each point of a Lorentz hypersurface M has a neighborhood (in M) on which is defined a vector field, usually denoted by ξ, consisting of space-like unit normals.

Let D be the usual (flat) connection on $\mathbf{L}^{n+1}$. The Levi-Civita connection ∇ on M is specified by the Gauss formula

$$D_X Y = \nabla_X Y + h(X, Y)\xi \tag{3.1}$$

where X and Y are tangent vector fields on M, and ξ is a local (space-like) unit normal field on M; if $x \in M$, then $(\nabla_X Y)_x$ is the tangential component of $(D_X Y)_x$ (cf. [KN, Vol. II, Chapter VII]).

In (3.1), h is a symmetric bilinear form, the *second fundamental form*. There is a field A of endomorphisms of $T(M)$, symmetric with respect to the Lorentz metric, such that

$$D_X \xi = -AX \tag{3.2}$$

and

$$\langle AX, Y \rangle = h(X, Y). \tag{3.3}$$

The curvature tensor R is related to the second fundamental tensor by the *equation of Gauss*:

$$R(X, Y) = AX \wedge AY \tag{3.4}$$

where $X \wedge Y$ denotes the endomorphism of the tangent space defined by

$$(X \wedge Y)Z = \langle Z, Y \rangle X - \langle Z, X \rangle Y.$$

The second fundamental tensor satisfies the *equation of Codazzi*:

$$(\nabla_X A)Y = (\nabla_Y A)X. \tag{3.5}$$

Finally, for each $x \in M$, $T_0(x) = \ker A_x = \{x \in T_x M\colon (AX)_x = 0\}$ is called the *relative nullity space at* x.

An important fact about R is the following.

PROPOSITION (3.6). $R \equiv 0$ *(that is,* $R(X, Y) = 0$ *for all* X, Y*) if and only if* M *is locally isomorphic with* $\mathbf{L}^n$.

PROOF. This is analogous to the proof in the Riemannian case (cf. [S, Vol. II, pp. 6–21]), with $\langle X^1, X^1 \rangle = -1$. Q.E.D.

Let us consider some examples of Lorentz hypersurfaces.

(3.7). $M \times \mathbf{R}$. Let M be a Riemannian manifold with metric g. If $\mathbf{R}$ is the real line, then define a Lorentz metric on the direct product $M \times \mathbf{R}$ as follows. A tangent vector to $M \times \mathbf{R}$ has the form $(X, a\partial/\partial t)$, where X is tangent to M and $a \in \mathbf{R}$. The metric is defined by

$$\langle (X, a), (Y, b) \rangle = g(X, Y) - ab.$$

Notice that if M lies in $\mathbf{E}^n$, then $M \times \mathbf{R}$ lies in $\mathbf{L}^{n+1}$.

(3.8). *Cylinder over a time-like curve*. Let $c(t)$ be a unit speed time-like curve in $\mathbf{L}^2$. Let $\mathbf{E}^1$ denote $\mathbf{R}$ with its usual (positive definite) metric $(dt)^2$, and let $\mathbf{L}^1$ denote $\mathbf{R}$ with the negative definite metric $-(dt)^2$. Then c is an isometric immersion: $\mathbf{L}^1 \to \mathbf{L}^2$, and, if the products are Riemannian direct products,

$$c \times \text{id}\colon \mathbf{L}^1 \times \mathbf{E}^1 \to \mathbf{L}^2 \times \mathbf{E}^1$$

defines an isometric immersion: $\mathbf{L}^2 \to \mathbf{L}^3$.

(3.9). *B-scroll*. Let $(x(s), F(s))$ be a Cartan-framed null curve in $\mathbf{L}^3$. The *B-scroll of* (x, F) is parametrized by $f(s, u) = x(s) + uB(s)$. Since $f_*(\partial/\partial u) = B(s)$ and $f_*(\partial/\partial s) = A(s) + uk_3(s)C(s)$, the metric at $f(s, u)$ has the matrix $\begin{bmatrix} a & -1 \\ -1 & a \end{bmatrix}$, of determinant -1, with respect to the basis $\{f_*(\partial/\partial s), f_*(\partial/\partial u)\}$. Thus, the metric is Lorentz.

There is a relationship between B-scrolls and flat Lorentz surfaces, which is described by the following theorem.

THEOREM (3.10). *The B-scroll is flat if and only if* $k_3(s) = 0$ *for all* s.

PROOF. If $k_3 \equiv 0$, let $\partial/\partial s = (1/\sqrt{2})\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\partial/\partial u = (1/\sqrt{2})\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ form a null frame for $\mathbf{L}^2$. Then f describes a local isometry between the B-scroll and $\mathbf{L}^2$ (see (3.9)).

Conversely, suppose the B-scroll is flat; then its curvature tensor is identically zero. Now, there are the equalities:

$$\left\langle \nabla_u \frac{\partial}{\partial u}, \frac{\partial}{\partial u} \right\rangle = \left\langle \nabla_s \frac{\partial}{\partial u}, \frac{\partial}{\partial u} \right\rangle = \left\langle \nabla_u \frac{\partial}{\partial u}, \frac{\partial}{\partial s} \right\rangle = 0,$$

$$\left\langle \nabla_s \frac{\partial}{\partial u}, \frac{\partial}{\partial s} \right\rangle = uk_3(s)^2.$$

That $\langle \partial/\partial u, \partial/\partial s \rangle = -1$ implies that $\nabla_s \partial/\partial u = -u(k_3(s))^2 \partial/\partial u$; and hence,

$$R\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial s} \right) \frac{\partial}{\partial u} = \nabla_u \nabla_s \frac{\partial}{\partial u} - \nabla_s \nabla_u \frac{\partial}{\partial u}$$

$$= -\nabla_u \left(uk_3(s)^2 \frac{\partial}{\partial u} \right) = -k_3(s)^2 \frac{\partial}{\partial u}.$$

If $R \equiv 0$ at all points of the B-scroll, then $k_3(s) = 0$ for all s. Q.E.D.

Thus, B-scrolls are Lorentz surfaces which are flat just in the case that the B-direction is parallel. Notice also in this case that $C(s)$ is a unit (space-like) normal which is parallel along the B-direction; it follows that $f_*(\partial/\partial u)$ generates the relative nullity space at each point $f(s, u)$ such that $k_2(s) \neq 0$ (cf. (3.11) below).

A Cartan-framed null curve whose B-direction is parallel will be called a *generalized null cubic* (*GNC*). The name can be justified as follows. By definition, the Frenet equations of a GNC $(x(s), F(s))$ have the form

$$\frac{dx}{ds} = A(s), \quad \frac{dA}{ds} = k_2(s)C(s), \quad \frac{dB}{ds} = 0, \quad \frac{dC}{ds} = k_2(s)B(s). \tag{3.11}$$

If $k_2(s) = 1$ for all s, then the curve $x(s)$ with initial condition $x(0) = 0$ and $F(0) = N$ is easily seen to be

$$x(s) = \begin{bmatrix} \frac{1}{2}\left(s + \frac{1}{6}s^3\right) \\ -\frac{1}{2}\left(s - \frac{1}{6}s^3\right) \\ \frac{1}{2}s^2 \end{bmatrix}. \tag{3.12}$$

Bonnor [**B**] calls such a curve a null cubic. It is from this that the present term "generalized null cubic" derives.

Example (2.6) is in fact a GNC. Therefore, Theorem (3.10) implies that

$$f(s, u) = \left(\tfrac{u}{2} + \tfrac{4}{3}s^3 + s,\ \tfrac{u}{2} + \tfrac{4}{3}s^3 - s,\ 2s^2\right) \tag{3.13}$$

describes a flat Lorentz surface in $\mathbf{L}^3$. (Here, of course, (s, u) are coordinates with respect to a null frame for $\mathbf{L}^2$, so

$$\left\langle \frac{\partial}{\partial s}, \frac{\partial}{\partial s} \right\rangle = \left\langle \frac{\partial}{\partial u}, \frac{\partial}{\partial u} \right\rangle = 0 \quad \text{and} \quad \left\langle \frac{\partial}{\partial u}, \frac{\partial}{\partial s} \right\rangle = -1;$$

the $\mathbf{L}^3$ coordinates are with respect to the standard orthonormal basis.)

PART II. COMPLETENESS OF THE RELATIVE NULLITY FOLIATION

4. Preliminary results. We consider isometric immersions: $\mathbf{L}^n \to \mathbf{L}^{n+1}$. It follows from the equation of Gauss (3.4) that the dimension $\nu_0(x)$ of the relative nullity space at each $x \in \mathbf{L}^n$ is $n - 1$ or n. If $\nu_0(x) = n$, then x is an *umbilic* (or *planar*) *point*. Define:

$$W = \{x \in \mathbf{L}^n\colon x \text{ is not an umbilic point}\}.$$

PROPOSITION (4.1). *W is an open subset of* $\mathbf{L}^n$.

PROOF. $W = \{x \in \mathbf{L}^n\colon A_x \equiv 0\}$. Q.E.D.

Let $x \in W$. Then $T_0(x)$ is an $(n-1)$-dimensional subspace of $\mathbf{L}^n_x = T_x(\mathbf{L}^n)$. Therefore, by Theorem (1.1), exactly one of the following statements is true:

(4.2) (i) $T_0(x)$ is an $(n-1)$-dimensional Lorentzian subspace,

(ii) $T_0(x)$ is an $(n-1)$-dimensional Euclidean subspace,

(iii) $T_0(x) \simeq E \oplus \operatorname{Span}\{\xi\}$ where E is an $(n-2)$-dimensional Euclidean subspace, ξ is a light-like vector, and $\langle Y, \xi\rangle = 0$ for all $Y \in E$.

In cases (i) and (ii) of (4.2) the metric inherited by $T_0(x)$ is nondegenerate (with index 1 and 0, respectively); in case (iii), the inherited metric is degenerate. We will say that the relative nullity (space) at x is *degenerate* or *nondegenerate* according to whether the inherited metric is degenerate or nondegenerate.

THEOREM (4.3). *Let $x \in W$. The following statements are equivalent*:

(i) *$T_0(x)$ is degenerate,*

(ii) *the image of A_x is a light line,*

(iii) $A_x^2 \equiv 0$.

PROOF. (i) $\Rightarrow$ (ii). $T_0(x) \simeq E \oplus \mathrm{Span}\{\xi\}$ per (4.2)(iii), so $\mathbf{L}_x^n = T_0(x) \oplus \mathrm{Span}\{\eta\}$, a direct but not orthogonal sum in which $\mathrm{Span}\{\xi, \eta\}$ is the Lorentz plane orthogonal to E in $\mathbf{L}_x^n$. Since $x \in W$, $A\eta \neq 0$. Clearly, $A\eta$ is orthogonal to E. But also $\langle A\eta, \xi\rangle = \langle \eta, A\xi\rangle = 0$. Since the only vectors in a Lorentz plane orthogonal to a nonzero light vector are linearly dependent vectors (see [G]), $\mathrm{Span}\{A\eta\} = \mathrm{Span}\{\xi\}$.

(ii) $\Rightarrow$ (iii). If $\eta \in \mathbf{L}_x^n$ and $\mathbf{L}_x^n = T_0(x) \oplus \mathrm{Span}\{\eta\}$, then $\langle A^2\eta, Y\rangle = \langle A\eta, AY\rangle = 0$ for all $Y \in T_0(x)$, and $\langle A^2\eta, \eta\rangle = \langle A\eta, A\eta\rangle = 0$. Thus, $A^2\eta = 0$.

(iii) $\Rightarrow$ (i). If $A_x^2 \equiv 0$, then the image of A_x lies in $T_0(x)$. Suppose the image of A_x is $\mathrm{Span}\{A\eta\}$ ($\eta \in \mathbf{L}_x^n$). If $Y \in T_0(x)$ then $\langle A\eta, Y\rangle = \langle \eta, AY\rangle = 0$. Thus, $A\eta$ is a vector in $T_0(x)$ which is orthogonal to every vector in $T_0(x)$. Q.E.D.

COROLLARY (4.4). *$G = \{x \in W$: $T_0(x)$ is nondegenerate$\}$ is an open subset of W (and hence of $\mathbf{L}^n$).*

PROOF. By Theorem (4.3), $G = \{x \in W: A_x^2 \not\equiv 0\}$. Q.E.D.

A *foliation* of an open subset of a manifold is an integrable differentiable distribution on the subset. A differentiable distribution–in particular, a foliation–is *totally geodesic* if $\nabla_X Y$ lies in the distribution whenever X and Y do. We remark that parallel transport along curves whose tangent vectors lie in a totally geodesic foliation preserves the foliation. The maximal connected integral submanifolds corresponding to a foliation according to the theorem of Frobenius are called the *leaves* of the foliation. A totally geodesic foliation is called *complete* if any affinely-parametrized geodesic which is tangent to the foliation can be extended to all values of the parameter and still lie in a leaf of the foliation.

On W, there is an $(n-1)$-dimensional distribution T_0: $x \to T_0(x)$, called the *relative nullity distribution*. The present purpose is to show that T_0 is a complete totally geodesic foliation of W. First, we will show that $T_0|G$ is a complete totally geodesic foliation of G, using methods appearing in [$\mathbf{N}_1$] and [$\mathbf{N}_2$], which derive from those in [A] and [F]. After showing that $W \setminus G$ is open in W (indeed, we shall ultimately see that $G = \varnothing$ or $G = W$), we will then show that $T_0|(W \setminus G)$ is a complete totally geodesic foliation of $W \setminus G$. It will then be apparent that T_0 is a complete totally geodesic foliation of W.

5. Nondegenerate relative nullities. In this section, we consider the restriction of the relative nullity distribution to G. Therefore, for each $x \in G$, $T_0(x)$ is a Lorentz or Euclidean $(n-1)$-dimensional subspace of $\mathbf{L}_x^n$, and there is a uniquely determined orthogonal complement to $T_0(x)$ in $\mathbf{L}_x^n$, which we will denote by $T_0^{\perp}(x)$.

PROPOSITION (5.1). *$T_0^\perp(x)$ is a nontrivial eigenspace for A_x.*

PROOF. $T_0^\perp(x)$ is generated by a nonzero vector Z such that $\langle Z, Z\rangle \neq 0$. If $Y \in T_0(x)$, then $0 = \langle Z, A_x Y\rangle = \langle A_x Z, Y\rangle$, so that there is some $\lambda(x) \neq 0$ such that $A_x Z = \lambda(x) Z$. Clearly, $\lambda(x)$ is independent of the choice of Z. Q.E.D.

REMARK. Since $\lambda(x)$ is the nonzero eigenvalue of A_x for each $x \in G$, the function λ is differentiable.

PROPOSITION (5.2). *T_0 and $T_0^\perp$ are differentiable distributions on G.*

PROOF. Lemma 1 of [$\mathbf{N}_1$]. Q.E.D.

PROPOSITION (5.3). *T_0 is integrable.*

PROOF. Lemma 2 of [$\mathbf{N}_1$]. Q.E.D.

Now suppose X is a vector field belonging to $T_0(x)$ and Z is a vector field belonging to $T_0^\perp$. The Codazzi equation (3.5) implies

$$\nabla_X(AZ) - A(\nabla_X Z) = \nabla_Z(AX) - A(\nabla_Z X). \tag{5.4}$$

Suppose a tangent vector U is decomposed into T_0- and $T_0^\perp$-components: $U = U_0 + U^\perp$. Then $AU = \lambda U^\perp$ and $(A - \lambda I)U = -\lambda U_0$. Since $AX = 0$ and $AZ = \lambda Z$, (5.4) implies

$$(X\lambda)Z + \lambda(\nabla_X Z)_0 = -\lambda(\nabla_Z X)^\perp. \tag{5.5}$$

Setting the T_0- and $T_0^\perp$-components on each side of (5.5) equal yields these useful equations:

$$(X \cdot \lambda)Z + \lambda(\nabla_Z X)^\perp = 0, \tag{5.6}$$

$$\lambda(\nabla_X Z)_0 = 0. \tag{5.7}$$

Since $\lambda \neq 0$, (5.7) implies that $\nabla_X Z \in T_0^\perp$. Now, if Y is a vector field belonging to T_0, then $\langle Y, Z\rangle = 0$, and so

$$0 = X \cdot \langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle = \langle \nabla_X Y, Z\rangle.$$

Thus, we have

PROPOSITION (5.8). *T_0 is a totally geodesic foliation of G.*

If $x \in G$, denote the leaf of the relative nullity foliation through x by $M_0(x)$. Proposition (5.8) implies that $M_0(x)$ is a piece of a hyperplane in $\mathbf{L}^n$ through x, since totally geodesic submanifolds of any $\mathbf{R}^m$ with the flat connection are pieces of hyperplanes.

Now let $x_0 \in G$ and suppose x_t is an affinely-parametrized geodesic which lies in $M_0(x_0)$ for $0 \leqslant t < b$. Since $\mathbf{L}^n$ is complete, x_t can be extended indefinitely in $\mathbf{L}^n$. To show that T_0 is a complete foliation of G, it suffices to show that for all t, $x_t \in M_0(x_0)$.

LEMMA (5.9). *If $x_b \in W$, then there exists a positive ε such that $x_t \in M_0(x_0) \leqslant G$ for $0 \leqslant t < b + \varepsilon$.*

PROOF. First, since T_0 and $T_0^\perp$ are parallel along the T_0-curve x_t, we may choose orthonormal parallel vector fields Y_t^i ($i = 1, \dots, n-1$) along x_t such that $T_0(x_t) = \text{Span}\{Y_t^1, \dots, Y_t^{n-1}\}$. Extend these parallelly (in $\mathbf{L}^n$) beyond $t = b$. Then for all t, $\text{Span}\{Y_t^1, \dots, Y_t^{n-1}\}$ is an $(n-1)$-dimensional subspace of $\mathbf{L}^n_{x_t}$. Moreover, $AY_b^j = \lim_{t\to b} AY_t^j = 0$ for $j = 1, \dots, n-1$, so $\text{Span}\{Y_b^1, \dots, Y_b^{n-1}\} \subseteq T_0(x_b)$. If $x_b \in W$, these spaces have the same dimension and so coincide. The metric is preserved by parallel transport, so $T_0(x_b)$ has the same metric as $T_0(x_t)$ for $t < b$. In this case, that metric is nondegenerate, so $x_b \in G$. By Corollary (4.4), x_t must lie in G for parameters t in some neighborhood of b. Now choose coordinates y^i about x_b such that the integral submanifolds of T_0 are given by y^n = constant. In particular, $M_0(x_b)$ is the slice $y^n = 0$. The geodesic x_t, for $t < b$ at least, lies in a slice $y^n = c$ which is $M_0(x_0)$ near x_b. But $c = \lim_{t\to b} x_t^n = x_b^n = 0$. Thus $M_0(x_0)$ and $M_0(x_b)$ must coincide, and $M_0(x_0)$ is an open hyperplane piece containing x_b. Q.E.D.

As a consequence of Lemma (5.9), it is imperative to show that $x_b \in W$ if x_t is the geodesic described above. To this end, let y be a point on the geodesic: $y = x_{t_1}$, $0 \leqslant t_1 < b$. Let X be an extension of the tangent vector field $\vec{x}_t$ to a T_0-field near y. If Z_t is a $T_0^\perp$-field along x_t, let Z be an extension of Z_t to a $T_0^\perp$-field near y. Define P: $T_0 \times T_0^\perp \to T_0^\perp$ near y by $P(X, Z) = (\nabla_Z X)^\perp$. (For the purposes of the definition, X and Z may be arbitrary T_0- and T_0-fields, respectively, near y.)

LEMMA (5.10). *If u is a point near y, then P gives a bilinear map*

$$P_u\colon T_0(u) \times T_0^\perp(u) \to T_0^\perp(u).$$

PROOF. Choose coordinates x^i near u so that

$$T_0 = \text{Span}\{\partial/\partial x^1, \dots, \partial/\partial x^{n-1}\}.$$

If $X = \sum_{j=1}^{n-1} X^j \partial/\partial x^j$, then

$$(\nabla_Z X)^\perp = \left[\sum_{j=1}^{n-1} (Z \cdot X^j)\left(\frac{\partial}{\partial x^j}\right) + \sum_{j=1}^{n-1} X^j\left(\nabla_Z \frac{\partial}{\partial x^j}\right)\right]^\perp$$

$$= \sum_{j=1}^{n-1} X^j\left(\nabla_Z \frac{\partial}{\partial x^j}\right)^\perp.$$

Thus, if $X_u = 0$ or $Z_u = 0$, then $(\nabla_Z X)_u^\perp = 0$. Q.E.D.

LEMMA (5.11). *If there is a vector field Z_t along x_t for $0 \leqslant t \leqslant b$ such that $Z_t \in T_0^\perp(x_t)$ and $\nabla_t Z_t = P(X, Z_t)_{x_t}$ for $0 \leqslant t < b$, then $x_b \in W$.*

PROOF. Let Y_t be a parallel field along x_t such that $Y_b \in T_0(x_b)$. For $t < b$,

$$\frac{d}{dt}\langle AZ_t, Y_t\rangle = \langle \nabla_t(\lambda Z_t), Y_t\rangle = \langle (X\lambda)Z + \lambda(\nabla_t Z_t), Y_t\rangle$$
$$= \langle (X\lambda)Z + \lambda(\nabla_Z X)^{\perp}, Y_t\rangle = 0$$

by (5.6). Since $\langle AZ_b, Y\rangle = \langle Z_b, AY_b\rangle = 0$, it follows that $\langle AZ_t, Y_t\rangle = 0$ for all $t \in [0, b]$, in particular for $t = 0$, but $\langle AZ, Y\rangle_{x_0} = \lambda(x_0)\langle Z, Y\rangle_{x_0}$, which, since $\lambda(x_0) \neq 0$, implies that $\langle Z_0, Y_0\rangle = 0$. Thus, Y_0 is orthogonal to $T_0^{\perp}(x_0)$; i.e., $Y_0 \in T_0(x_0)$. Therefore, the parallel transport isomorphism maps the $(n-1)$-dimensional $T_0(x_0)$ onto $T_0(x_b)$, so $\dim T_0(x_b) = n - 1$; $x_b \in W$. Q.E.D.

LEMMA (5.12). *A vector field Z_t exists along x_t which satisfies the hypotheses of Lemma* (5.11).

PROOF. Suppose Z_t is a solution to the differential equation $\nabla_t Z_t = P_{x_t}(\vec{x}_t, Z_t)$ along x_t satisfying a $T_0^{\perp}$ initial condition at $t = 0$. If $y = x_{t_1}$ is a point on the geodesic under consideration, then let X and Z be the extensions of $\vec{x}_t$ and Z_t near y mentioned above. Then at each y on the geodesic, the equation $\nabla_X Z = P(X, Z)$ holds. Therefore,

$$\nabla_X^2 Z = \nabla_X(P(X, Z)) = \nabla_X(\nabla_Z X)^{\perp} = (\nabla_X \nabla_Z X)^{\perp};$$

the last equality holds because $X \in T_0$ and T_0 is totally geodesic. Now, since $R \equiv 0$ and there is no torsion ($[X, Z] = \nabla_X Z - \nabla_Z X$), we get

$$\nabla_X^2 Z = (\nabla_Z \nabla_X X)^{\perp} + (\nabla_{\nabla_X Z} X)^{\perp} - (\nabla_{\nabla_Z X} X)^{\perp}. \tag{5.13}$$

Since $X \in T_0$, $\nabla_X X \in T_0$, and in particular $(\nabla_X X)_y = (\nabla_t \vec{x}_t)_{t_1} = 0$. So if $Y \in T_0^{\perp}$,

$$\langle \nabla_Z \nabla_X X, Y\rangle_y = Z\cdot\langle \nabla_X X, Y\rangle_y - \langle (\nabla_X X)_y, \nabla_Z Y\rangle = 0.$$

Thus, along the geodesic, $(\nabla_Z \nabla_X X)^{\perp} = 0$. Since $X \in T_0$ and $Z \in T_0^{\perp}$, $\nabla_X Z \in T_0^{\perp}$ (see (5.7)) and so by definition $(\nabla_{\nabla_X Z} X)^{\perp} = P(X, \nabla_X Z)$. Finally,

$$(\nabla_{\nabla_Z X} X)^{\perp} = (\nabla_{(\nabla_Z X)_0 + (\nabla_Z X)^{\perp}} X)^{\perp}$$
$$= (\nabla_{(\nabla_Z X)_0} X)^{\perp} + (\nabla_{(\nabla_Z X)^{\perp}} X)^{\perp}$$
$$= (\nabla_{(\nabla_Z X)} X)^{\perp} = P(X, P(X, Z)).$$

With this information, (5.13) becomes (along the geodesic)

$$\nabla_X^2 Z = P(X, \nabla_X Z) - P(X, P(X, Z)) = P(X, \nabla_X Z - P(X, Z)) = 0.$$

Now, since P: $T_0 \times T_0^{\perp} \to T_0^{\perp}$ is bilinear, the linear differential equation $\nabla_t Z_t = P(\vec{x}_t, Z_t)$ admits a unique solution Z_t, $0 \leqslant t < b$, corresponding to any initial condition $Z_0 \in T_0^{\perp}(x_0)$. We have just seen that such a solution must satisfy $\nabla_t^2 Z_t = 0$. If $\mathfrak{Z}$ is a parallel $T_0^{\perp}$-field along x_t, $0 \leqslant t < b$, then, for some smooth Φ: $\mathbf{R} \to \mathbf{R}$, we have $Z_t = \Phi(t)\mathfrak{Z}$ and $d^2\Phi/dt^2 = 0$. Therefore, $\Phi(t) = \alpha t + \beta$, where the constants α and β are determined by

$$Z_0 = \beta \mathfrak{Z},$$
$$\alpha \mathfrak{Z} = (\nabla_t Z_t)_{t=0} = P(\vec{x}_t, Z_t)|_{t=0} = P(\vec{x}_0, \beta \mathfrak{Z}).$$

Since $\mathfrak{Z}$ can be extended as a parallel vector field along x_t to $t = b$ and beyond, the vector field $Z_t = (\alpha t + \beta)\mathfrak{Z}$ can be extended to beyond $t = b$ as well. If $0 \leqslant t < b$, then $Z_t \in T_0^{\perp}(x_t)$. Q.E.D.

THEOREM (5.14). *T_0 is a complete totally geodesic foliation of G.*

PROOF. There is to show that if $x_0 \in G$ and x_t is a geodesic emanating from x_0 in a T_0 direction and extended indefinitely in the complete $\mathbf{L}^n$, then $x_t \in M_0(x_0) \subseteq G$ for all t. Let b be defined by

$$b = \sup\{u\colon x_t \in M_0(x_0) \text{ for } 0 \leqslant t < u\}.$$

Then $b > 0$. Suppose b is finite. Then, by Lemma (5.12), choose a vector field along x_t such that, for $0 \leqslant t < b$, $Z_t \in T_0^{\perp}(x_t)$ and $\nabla_t Z_t = P(\vec{x}_t, Z_t)$. By Lemma (5.11), $x_b \in W$. But now Lemma (5.9) implies that for some positive ε, $x_t \in M_0(x_0)$ for $0 \leqslant t < b + \varepsilon$. This contradicts the nature of b as a supremum. Therefore, b is infinite. Q.E.D.

COROLLARY (5.15). *For each $x \in G$, $M_0(x)$ is a hyperplane contained in G.*

COROLLARY (5.16). *The open set G is a union of parallel hyperplanes* (*with metric of the same signature*).

If $x \in G$ and/or $x \in W$, let G_x (respectively, W_x) denote the component of x in G (respectively, in W).

PROPOSITION (5.17). *If $x \in G$, then $G_x = W_x$.*

PROOF. Suppose x_t is a curve from $x_0 = x$ to $x_b = z$ in W_x such that $x_t \in G_x$ for each $t \in [0, b)$. Then (5.16) implies that $\{T_0(x)\colon t \in [0, b)\}$ is a family of parallel nondegenerate tangent hyperplanes. Therefore, there are vector fields Y_t^i $(i = 1, \ldots, n-1)$ along x_t, $0 \leqslant t < b$, such that, for $t \in [0, b)$,

(i) $T_0(x_t) = \operatorname{Span}\{Y_t^1, \ldots, Y_t^{n-1}\}$,
(ii) $\langle Y_t^i, Y_t^j\rangle = \delta_i^j$ (with $\delta_1^1 = -1$ if T_0's are Lorentz),
(iii) $\nabla_t Y_t^i = 0$, $i = 1, \ldots, n-1$.

Extend each Y_t^i parallelly along x_t. Since $\langle Y_t^i, Y_t^j\rangle$ is preserved, $\operatorname{Span}\{Y_b^1, \ldots, Y_b^{n-1}\}$ is a nondegenerate tangent hyperplane through z. Also $AY_b^i = \lim_{t\to b} AY_t^i = 0$, so this hyperplane lies in $T_0(z)$. Since $z \in W_x$, $T_0(z)$ has dimension $n-1$, and therefore coincides with the nondegenerate hyperplane Span$\{Y_b^1, \ldots, Y_b^{n-1}\}$. As a result, $z \in G_x$. It follows from this argument that if x_t is any curve in W_x emanating from $x \in G_x$, then $\sup\{u: x_t \in G_x \text{ for } t \in [0, u)\}$ is not finite. Q.E.D.

6. Degenerate relative nullities. Now we consider $W \setminus G$, the subset of points $y \in W$ such that $T_0(y)$ carries a nondegenerate metric. By (4.2)(iii), we have at each such y a decomposition

$$T_0(y) = E(Y) \oplus \operatorname{Span}\{\xi\}$$

where ξ is a light vector and $E(y)$ is an $(n-2)$-dimensional Euclidean subspace of $\mathbf{L}_y^n$ orthogonal to ξ. Notice that although the light line in $T_0(y)$ is unique, the Euclidean "complement" $E(y)$ is not. Since $\mathbf{L}^n$ is locally connected, (5.17) implies

PROPOSITION (6.1). *$W \setminus G$ is open.*

Now define a distribution on the open set $W \setminus G$ by $y \mapsto T_0(y)$. It is, of course, the restriction of the relative nullity foliation to the open set $W \setminus G$.

PROPOSITION (6.2). *T_0 is a differentiable distribution on $W \setminus G$.*

PROOF. Consider a point y of $W \setminus G$, and select a local light field L at y–i.e., a vector field L in a neighborhood of y such that $\langle L, L\rangle = 0$–which satisfies $(AL)_y \neq 0$. Then $AL \neq 0$ near y and, by (4.3)(ii), AL is a local light field at y.

Indeed, for those x near y, AL_x lies in the light line of $T_0(x)$, and so $AL_x \perp T_0(x)$. If also $AL_x \perp L_x$, then $AL_x \perp \mathbf{L}_x^n$, implying $AL_x = 0$, a contradiction if x is close to y. So $\langle AL_x, L_x\rangle \neq 0$. Now one can compute directly that $\{AL + L, AL - L\}$ is an orthogonal set of two vectors, one space-like and one time-like, which therefore spans a Lorentz plane $\mathbf{L}^2(x)$ in $\mathbf{L}_x^n$.

Define $\mathbf{E}^{n-2}(x)$ to be the orthogonal complement of $\mathbf{L}^2(x)$ in $\mathbf{L}_x^n$. Then $\mathbf{E}^{n-2}(x) \subseteq T_0(x)$. For, if $Y \in \mathbf{E}^{n-2}(x)$, then $\langle AY, L\rangle = \langle Y, AL\rangle = 0$ and, by (4.3)(iii), $\langle AY, AL\rangle = \langle Y, A^2L\rangle = 0$, so A_x maps $\mathbf{E}^{n-2}(x)$ into itself. But the image of A_x is Span$\{AL_x\}$, so $AY = 0$ for all $Y \in \mathbf{E}^{n-2}(x)$.

Thus, $\mathbf{E}^{n-2}(x)$ and AL_x play the role of E and ξ for $T_0(x)$ as described by (4.2). Note that $\mathbf{E}^{n-2}(x)$ depends on the choice of the local light field L. L and AL are differentiable in a neighborhood of y. It suffices then to show that $x \mapsto \mathbf{E}^{n-2}(x)$ defines a differentiable distribution near y.

Normalize $\{AL + L, AL - L\}$ at each x near y to give an orthonormal basis $\{T_x, S_x\}$ for $\mathbf{L}_x^2$, where T and S are differentiable unit time-like and space-like vector fields, respectively. Next, suppose $\mathbf{E}^{n-2}(y) = \operatorname{Span}\{Y_y^1, \ldots, Y_y^{n-2}\}$. For $j = 1, \ldots, n-2$, extend Y_y^j to a vector field Y^j in a neighborhood of y, and (in a possibly smaller neighborhood of y) define Z^j by

$$Z^j = Y^j - \langle Y^j, S\rangle S + \langle Y^j, T\rangle T.$$

Clearly, $\langle Z^j, S\rangle = \langle Z^j, T\rangle = 0$, so $Z^j_x \in \mathbf{E}^{n-2}(x)$, for each j. Since $Z^j_y = Y^j_y$, $\{Z^j\}_{j=1}^{n-2}$ is linearly independent at y, and hence also at points near y. For those points x, $x \mapsto \mathbf{E}^{n-2}(x)$ is thus a differentiable distribution. Q.E.D.

PROPOSITION (6.3). *T_0 is integrable; so $T_0|(W\setminus G)$ foliates $W\setminus G$.*

PROPOSITION (6.4). *The light lines in the relative nullity foliation are parallel along any T_0-direction.*

PROOF. If $Y \in T_0$, then it follows from the Codazzi equation (3.5) that $\nabla_Y(AL)y = -A([L, Y])_y$ lies in image $A_y = \mathrm{Span}\{AL_y\}$ if $y \in W\setminus G$. Q.E.D.

PROPOSITION (6.5). *T_0 is a totally geodesic foliation of $W\setminus G$.*

PROOF. The following conventions will be in order for a point $y \in W\setminus G$. The constructions are essentially those of the proof of Proposition (6.2). L will be a local light field at y, so that AL generates the light lines in the relative nullity spaces. $\mathbf{E}^{n-2}$ will denote the orthogonal complement of $\mathrm{Span}\{L, AL\}$ and have orthonormal basis $\{Z^j\}_{j=1}^{n-2}$. If $U \in \mathbf{L}^n_y$, then the following formula holds.

$$U = \left(\sum_{j=1}^{n-2} \langle Z^j_y, U\rangle Z^j_y\right) + \frac{1}{\lambda(y)}\left(\langle U, L_y\rangle AL_y + \langle U, AL_y\rangle L_y\right) \tag{6.6}$$

where $\lambda(y) = \langle AL_y, L_y\rangle \neq 0$.

Suppose X and Y are T_0 vector fields near $y \in W\setminus G$; then, $\langle Y, AL\rangle = 0$ and

$$\begin{aligned} 0 = X\cdot\langle Y, AL\rangle &= \langle \nabla_X Y, AL\rangle + \langle Y, \nabla_X(AL)\rangle \\ &= \langle \nabla_X Y, AL\rangle + \langle Y, \nabla_L(AX)\rangle + \langle Y, A([L, X])\rangle \\ &= \langle \nabla_X Y, AL\rangle + \langle AY, A([L, X])\rangle = \langle \nabla_X Y, AL\rangle. \end{aligned}$$

By (6.6), this implies that the L-component of $(\nabla_X Y)_y = 0$. That is, $(\nabla_X Y)_y \in T_0(y)$. Q.E.D.

Proposition (6.5) implies that the leaf $M_0(y)$ of the relative nullity foliation through $y \in W\setminus G$ is an open piece of a hyperplane through y. This hyperplane carries a degenerate metric. Let x_0 be a fixed point of $W\setminus G$ and suppose x_t is an affinely-parametrized geodesic in $M_0(x_0)$, where $0 \leqslant t < b$. This geodesic can be extended to all values of the parameter in the complete space $\mathbf{L}^n$. To show that T_0 furnishes a complete foliation of $W\setminus G$, we need, according to Lemma (5.9) (whose proof, except to show that $M_0(x_0) \subseteq G$, is independent of the specific metric), to demonstrate that $x_b \in W$.

Suppose Ω generates the light line in $T_0(x_0)$. Extend Ω parallelly along $M_0(x_0)$. By Proposition (6.4), Ω_y generates the light line in $T_0(y)$ for each $y \in M_0(x_0)$. Now choose a light vector $L \in \mathbf{L}^n_{x_0}$ such that $\langle L, \Omega_{x_0}\rangle = -1$ and $A_{x_0}L \neq 0$. Then $A_{x_0}L = \rho_0\Omega_{x_0}$ with $\rho_0 \neq 0$. Extend L parallelly along $M_0(x_0)$; then AL_y is a multiple of Ω_y for each $y \in M_0(x_0)$. Along x_t for $t \in [0, b)$, $(AL)_{x_t} = \rho(t)\Omega_t$ where $\Omega_t = \Omega_{x_t}$ and $\rho(0) = \rho_0$. We can extend L and Ω parallelly along x_t for all t as light vectors satisfying $\langle L, \Omega\rangle = -1$. If we define ρ by $\rho(t) = -\langle AL, L\rangle_{x_t}$ for all t, then $(AL)_{x_t} = \rho(t)\Omega_t$ for $0 \leqslant t < b$, but where $\rho(t)$ is a smoothly defined function for all t.

In what follows, when evaluating certain quantities at a fixed point $y = x_{t_1}$ on the geodesic, we will extend $\vec{x}_t$ and L, respectively, to a T_0-field X and (e.g., by parallel extension in $\mathbf{L}^n$) a light field L, respectively, in a neighborhood N_y of y.

At a point y on the geodesic, we have, from the Codazzi equation,

$$\nabla_L(AX) - A(\nabla_L X) = \nabla_X(AL) - A(\nabla_X L). \tag{6.7}$$

We wish to evaluate (6.7) at y. Since $X \in T_0$, $AX = 0$ in N_y, and hence $(\nabla_L(AX))_y = 0$. Also, L is parallel along $M_0(x_0) = M_0(y)$, so that $X_y \in T_0(y)$ implies that $A(\nabla_X L)_y = A(0) = 0$. Thus, at y, (6.7) becomes

$$(\nabla_X(AL))_y + A(\nabla_L X)_y = 0. \tag{6.8}$$

Now, if Z is a vector field, then $\nabla_X Z$ depends on the behavior of Z on an integral curve of X through the point at which $\nabla_X Z$ is evaluated. Here, the integral curve of X through y is the geodesic x_t, along which $(AL)_{x_t} = \rho(t)\Omega_t$ for $t \in [0, b)$. Since Ω_t is parallel along x_t, we have

$$(\nabla_X(AL))_y = (\nabla_t(\rho(t)\Omega_t))_{t=t_1} = \rho'(t_1)\Omega_{t_1}. \tag{6.9}$$

Next, $\langle L, \Omega\rangle_y = -1$, so if $U \in \mathbf{L}^n_y$, then

$$U = U^\perp - \langle U, L_y\rangle\Omega_y - \langle U, \Omega_y\rangle L_y$$

where $U^\perp \in (\mathrm{Span}\{L_y, \Omega_y\})^\perp$ (cf. (6.6)). Since $U^\perp, \Omega_y \in T_0(y)$,

$$AU = -\langle U, \Omega_y\rangle AL_y, \tag{6.10}$$

and, in particular,

$$A(\nabla_L X)_y = -\langle(\nabla_L X)_y, \Omega_y\rangle\rho(t_1)\Omega_{t_1}. \tag{6.11}$$

Equations (6.8), (6.9) and (6.11) yield the following equation for $0 \leqslant t < b$:

$$[\rho'(t) - \rho(t)\langle(\nabla_L X)_{x_t}, \Omega_t\rangle]\Omega_t = 0. \tag{6.12}$$

For $0 \leqslant t < b$, define $\psi(t)$ by

$$\psi(t) = \langle(\nabla_L X)_{x_t}, \Omega_t\rangle.$$

Since ρ is a well-defined smooth function, equation (6.12) implies that ψ is a well-defined smooth function on $[0, b)$. The next step is to investigate the behavior of $\psi(t)$ along the geodesic x_t for $t \in [0, b)$. First, extend Ω (which so far has been defined on $M_0(x_0)$ as well as along all of x_t) to a neighborhood of $y = x_{t_1}$ in $\mathbf{L}^n$ as follows. Define Ω by

$$\Omega = \frac{-AL}{\langle AL, L\rangle};$$

this extends Ω near y, and is well defined since $\langle AL, L\rangle$ is nonzero at, hence near, y. Points near y lie in $W \setminus G$, so Ω generates the degenerate direction (or light line) of the relative nullity space at those points.

Now,

$$\frac{d\psi}{dt} = X \cdot \psi = X \cdot \langle \nabla_L X, \Omega\rangle = \langle \nabla_X \nabla_L X, \Omega\rangle + \langle \nabla_L X, \nabla_X \Omega\rangle. \quad (6.13)$$

Since $R \equiv 0$ and there is no torsion, equation (6.13) is equivalent to

$$\frac{d\psi}{dt} = \langle \nabla_L \nabla_X X, \Omega\rangle - \langle \nabla_{\nabla_L X} X, \Omega\rangle + \langle \nabla_{\nabla_X L} X, \Omega\rangle + \langle \nabla_L X, \nabla_X \Omega\rangle. \quad (6.14)$$

The last two terms vanish because $\nabla_X L = \nabla_X \Omega = 0$ along x_t. Next, $(\nabla_X X)_y = (\nabla_t \vec{x}_t)_{t_1} = 0$, so

$$\langle \nabla_L \nabla_X X, \Omega\rangle_y = L_y \langle \nabla_X X, \Omega\rangle - \langle (\nabla_X X)_y, (\nabla_L \Omega)_y\rangle = L_y \langle \nabla_X X, \Omega\rangle.$$

Since T_0 is totally geodesic, $\nabla_X X \in T_0$. Therefore, $\langle \nabla_X X, \Omega\rangle$, and hence $L_y \langle \nabla_X X, \Omega\rangle$, vanishes. Finally, consider the algebraic decomposition

$$(\nabla_L X)_y = (\nabla_L X)_0 - \langle (\nabla_L X)_y, \Omega_y\rangle L_y$$

where $(\nabla_L X)_0 \in T_0(y)$ (cf. (6.6)). T_0 is totally geodesic and Ω_y generates the degenerate direction of $T_0(y)$, so $\langle \nabla_{(\nabla_L X)_0} X, \Omega_y\rangle = 0$. Therefore,

$$\begin{aligned}\langle \nabla_{\nabla_L X} X, \Omega\rangle_y &= -\langle \nabla_{\langle (\nabla_L X)_y, \Omega_y\rangle L_y} X, \Omega\rangle_y \\ &= -\langle (\nabla_L X)_y, \Omega_y\rangle \langle \nabla_L X, \Omega\rangle_y = -\psi(t_1)^2. \quad (6.15)\end{aligned}$$

Thus, (6.14) becomes

$$\frac{d\psi}{dt} = \psi(t)^2 \quad (6.16)$$

for $0 \leqslant t < b$.

If $\psi(a) = 0$ for some $a \in [0, b)$, then $\psi \equiv 0$ or $[0, b)$. Otherwise, it is easily seen that

$$\psi(t) = \frac{\psi(0)}{1 - \psi(0)t} \quad (6.17)$$

for $0 \leqslant t < b$.

Now, to show that $x_b \in W$ is to show that $(AL)_{x_b} \neq 0$. But

$$(AL)_{x_b} = \lim_{t \to b} (AL)_{x_t} = \lim_{t \to b} \rho(t)\Omega_t = \left(\lim_{t \to b} \rho(t)\right)\Omega_b.$$

Recall that ρ is a well-defined smooth function for all t, so $\lim_{t\to b}\rho(t)$ exists and equals $\rho(b)$. Equation (6.12) implies

$$\rho(t) = \rho_0 \exp\left(\int_0^t \psi(s)\,ds\right), \qquad 0 \leqslant t < b. \tag{6.18}$$

If $\psi \equiv 0$, then $\rho(t) = \rho_0$, so $(AL)_{x_b} = \rho_0\Omega_b \neq 0$. If (6.17) holds, then (6.18) implies that

$$\rho(t) = \frac{\rho_0}{|1 - \psi(0)\cdot t|}, \qquad 0 \leqslant t < b.$$

Because ρ is smoothly defined for all t, $b \neq 1/\psi(0)$ and $\rho(b) = \rho_0/|1 - \psi(0)b| \neq 0$, implying that $x_b \in W$.

As in the proof of Theorem (5.14), that $x_b \in W$ implies that

$$\sup\{u\colon x_t \in M_0(x_0) \text{ for } 0 \leqslant t < u\}$$

is infinite, and gives the following results.

THEOREM (6.19). *T_0 is a complete totally geodesic foliation of $W \setminus G$.*

COROLLARY (6.20). *For each $x \in W \setminus G$, $M_0(x)$ is a hyperplane contained in $W \setminus G$.*

COROLLARY (6.21). *$W \setminus G$ is a union of parallel hyperplanes through points of an open set.*

Since G and $W \setminus G$ are disjoint open sets comprising W, Theorems (5.14) and (6.19) yield the following pivotal result.

THEOREM (6.22). *T_0 is a complete totally geodesic foliation of W.*

In a similar manner, their corollaries combine to give this description of the umbilic-free set W.

THEOREM (6.23). *The open set W is the union of a family of parallel hyperplanes.*

Of course, each such hyperplane is a leaf of the relative nullity foliation, and parallel hyperplanes must have the same inherited metric:

THEOREM (6.24). *Either $G = \varnothing$ or $G = W$.*

PART III. THE CLASSIFICATION THEOREMS

7. Preliminary remarks. In this part of the paper, we classify, up to a proper motion of $\mathbf{L}^{n+1}$, all isometric immersions: $\mathbf{L}^n \to \mathbf{L}^{n+1}$. To do this, we prove an appropriate "Moore's Lemma", use the results of Part II to divide our discussion into two cases ($G = \varnothing$ and $G = W$), and then, for each case, describe the structure induced on $\mathbf{L}^n$ by T_0. To obtain our desired description of the immersion f: $\mathbf{L}^n \to \mathbf{L}^{n+1}$, it then remains to see how the structure induced by T_0 is mapped by f into $\mathbf{L}^{n+1}$. To the degenerate case ($G = \varnothing$), the results of Part I will be pertinent.

For the remainder of this section, let $M = M_0 \times M_1$ be a product of connected manifolds-with-connection. Hence, if X is a tangent vector field on one factor and Y is a tangent vector field on the other, then $\nabla_X Y = 0$. Suppose further that M has a nondegenerate metric compatible with its connection. Let f: $M \to \mathbf{R}^N$ be an isometric immersion of M into a real vector space with nondegenerate metric and the usual flat connection. We assume there is a point $(x_0, y_0) \in M$ with $f(x_0, y_0) = 0 \in \mathbf{R}^N$. Define f_i: $M_i \to \mathbf{R}^N$ $(i = 0, 1)$ as follows:

$$\text{if } x \in M_0, \quad f_0(x) = f(x, y_0);$$
$$\text{if } y \in M_1, \quad f_1(y) = f(x_0, y).$$

If X, Y are tangent vector fields on M, then let

$$D_X f_* Y = f_*(\nabla_X Y) + \alpha(X, Y)$$

describe the decomposition of $D_X f_* Y$ into tangential and normal components. (Here, D is the covariant differentiation with respect to the flat connection on $\mathbf{R}^N$; ∇ is the covariant differentiation on M.)

LEMMA (7.1) ("MOORE'S LEMMA" [M]). *If, everywhere on M, $\alpha(X, Y) = 0$ whenever X is tangent to M_0 and Y is tangent to M_1, then $f(x, y) = f_0(x) + f_1(y)$ for every $(x, y) \in M$.*

PROOF. First, notice that if X is tangent to M_0 and Y is tangent to M_1, the hypothesis implies that $D_X f_* Y = 0$. Therefore,

(7.2) If $y \in M_1$, $Y \in T_y M_1$, and x_t is a curve in M, then the vector field $f_{*(x_t, y)}(0, Y)$ along $f(x_t, y)$ is parallel in $\mathbf{R}^N$.

Let x_t be a curve from x_0 to x in M_0 and let y_t be a curve from y_0 to y in M_1. Consider the curves $f(x, y_t)$ and $f_0(x) + f_1(y_t)$ in $\mathbf{R}^N$. If $t = 0$, they both take the value $f(x, y_0)$. On the other hand,

$$\frac{d}{dt}\left[f_0(x) + f_1(y_t)\right] = \frac{d}{dt} f(x_0, y_t) = f_{*(x_0, y_t)}(0, \vec{y}_t)$$

and

$$\frac{d}{dt} f(x, y_t) = f_{*(x, y)}(0, \vec{y}_t).$$

By (7.2) the two curves have the same tangent vectors. They agree at $t = 0$ and hence coincide. Q.E.D.

8. The nondegenerate case ($G = W$). Fix an origin 0 of $\mathbf{L}^n$ in W, and let M_0 be the hyperplane-leaf through 0. For a unit vector Z in $T_0^{\perp}(0)$, denote Span$\{Z\}$ by tZ $(t \in \mathbf{R})$. Then

$$\mathbf{L}^n = M_0 \times tZ = \{(x, t): x \in M_0, t \in \mathbf{R}\} \tag{8.1}$$

describes $\mathbf{L}^n$ both as a direct product of vector spaces and a direct product of pseudo-Riemannian manifolds. If $z = (x, t) \in W$, then $M_0(z) = \{(y, t): y \in M_0\}$, since the leaves of T_0 are parallel hyperplanes. If $z = (x, t) \notin W$, then *define* $M_0(z) = \{(y, t): y \in M_0\}$ and $T_0(z) = T_z(M_0(z))$. In either case, $\partial/\partial t$ generates the orthogonal complement of $T_0(z)$.

We may make the following assertions for *all* $z \in \mathbf{L}^n$. If $X \in T_0(z)$, then $AX = 0$; $A(\partial/\partial t) = \lambda(z)\partial/\partial t$, with $\lambda(z) \neq 0$ just in case that $z \in W$. The Codazzi equation (3.5) gives, in analogy with (5.6), $(X \cdot \lambda)(\partial/\partial t) + \lambda(\nabla_t X)^\perp = 0$. Since the T_0 planes are, or are chosen to be, parallel, $(\nabla_t X)^\perp = 0$. Therefore, $X \cdot \lambda = 0$.

Proposition (8.2). *λ is constant on each $M_0(z)$; hence, λ is a function of t.*

Now consider the immersion $f: \mathbf{L}^n \to \mathbf{L}^{n+1}$. Assume $f(0) = 0$. Take M_0 as in (8.1), and define $M_1 = tZ$. $M = \mathbf{L}^n$ and $\mathbf{R}^N = \mathbf{L}^{n+1}$ clearly have nondegenerate metrics. If $X \in T_0(z) = T_z(M_0)$ and $Y \in T_z M_1$ at $z \in \mathbf{L}^n$, then $\alpha(X, Y) = h(X, Y)\xi = \langle AX, Y\rangle\xi = \langle 0, Y\rangle\xi = 0$, where ξ is a unit normal field on $f(\mathbf{L}^n)$. Therefore, writing $f_1(t) = f_1(tZ)$, we have according to "Moore's Lemma"

$$f(x, t) = f_0(x) + f_1(t), \qquad (x, t) \in M_0 \times \mathbf{R}. \tag{8.3}$$

Proposition (8.4). *$f_0 = f|M_0$ is an isometry of M_0 onto an $(n - 1)$-plane in $\mathbf{L}^{n+1}$.*

Proof. Let $x_s = (x_s, 0)$ be a geodesic line emanating from 0 in the hyperplane M_0. (So $x_s = s\vec{x}_0$.) Then

$$D_s \overrightarrow{f(x_s, 0)} = D_s f_*(\vec{x}_s, 0) = f_*(\nabla_s \vec{x}_s) + \langle A\vec{x}_s, \vec{x}_s\rangle\xi = 0.$$

Thus $f(s\vec{x}_0) = sf_*(\vec{x}_0, 0)$, and $f(M_0)$ is an $(n - 1)$-plane spanned by $f_{*0}T_0(M)$. Q.E.D.

Corollary (8.5). *If $z = (x, t) \in \mathbf{L}^n$, $f(M_0(z)) = f_0(M_0) + f_1(t)$. Hence, $\{f(M_0(z)): z \in \mathbf{L}^n\}$ is a family of parallel $(n - 1)$-planes in $\mathbf{L}^{n+1}$.*

For $z \in \mathbf{L}^n$, let $[f(M_0(z))]^\perp$ be the plane in $\mathbf{L}^{n+1}$ at $f(z)$ orthogonal to $f(M_0(z))$. f is an isometry, so the planes $[f(M_0(z))]^\perp$ are parallel, and $f_1(t)$ lies in the plane $[f(M_0)]^\perp$. Writing $\mathbf{L}^{n+1}$ as the direct product $f(M_0) \times [f(M_0)]^\perp$, we may now write

$$f(x, t) = (f_0(x), f_1(t)) \in f(M_0) \times [f(M_0)]^\perp \tag{8.6}$$

for $(x, t) \in M_0 \times \mathbf{R} = \mathbf{L}^n$. Since the relative nullities are nondegenerate, there are, by Theorem (1.1), only two cases to consider.

(i) $M_0 \approx \mathbf{E}^{n-1}$. Then $f(M_0)^\perp$ is a Lorentz plane $\mathbf{L}^2$ in which $f_1(t)$ is a unit speed time-like curve $c(\langle dc/dt, dc/dt\rangle = -1)$. Since $f_0(s\vec{x}_0) = sf_{*0}(\vec{x}_0, 0)$ for $\vec{x}_0 \in T_0(0) = M_0$, we may treat f_0 as the identity map of $\mathbf{E}^{n-1}$ onto itself (up to a proper motion of $\mathbf{L}^{n+1}$).

(ii) $M_0 \approx \mathbf{L}^{n-1}$. Then $f(M_0)^\perp$ is a Euclidean plane $\mathbf{E}^2$ in which $f_1(t)$ is a unit-speed curve $c(\langle dc/dt, dc/dt\rangle = 1)$. In analogy with (i), we may treat f_0 as the identity map of $\mathbf{L}^{n-1}$ onto itself (up to a proper motion of $\mathbf{L}^{n+1}$).

(8.7). FIRST CLASSIFICATION THEOREM. *Up to a proper motion of* $\mathbf{L}^{n+1}$, *the isometric immersions*: $\mathbf{L}^n \to \mathbf{L}^{n+1}$ *with nondegenerate relative nullities have one of the following forms*:

(i) $\mathbf{E}^{n-1} \times \mathbf{L}^1 \to^{\mathrm{id}\times c} \mathbf{E}^{n-1} \times \mathbf{L}^2$, *with c a unit-speed time-like curve in* $\mathbf{L}^2$,

(ii) $\mathbf{L}^{n-1} \times \mathbf{E}^1 \to^{\mathrm{id}\times c} \mathbf{L}^{n-1} \times \mathbf{E}^2$, *with c a unit-speed Euclidean plane curve.* (*All products are orthogonal.*)

The surfaces described by Theorem (8.7) are cylinders over plane curves as mentioned in the example (3.8) and as in the Hartman-Nirenberg theorem (see (0.1)). Indeed the proof of (8.7) also shows that all isometric immersions: $\mathbf{E}^n \to \mathbf{L}^{n+1}$ have the form

$$\mathrm{id} \times c\colon \mathbf{E}^{n-1} \times \mathbf{E}^1 \to \mathbf{E}^{n-1} \times \mathbf{L}^2$$

where c is a unit-speed space-like curve in $\mathbf{L}^2$.

Furthermore, there is a relationship between the nonzero eigenvalue of the second fundamental tensor and the curvature of the plane curve c in each of those theorems, which we will briefly outline in the case of Theorem (8.7).

In case (ii) of that theorem, at each point $z \in \mathbf{L}^n$, $[f(M_0(z))]^\perp$ is spanned orthonormally by $f_{*z}\partial/\partial t = d/dt(f_1(t))$ and ξ_z (where ξ is a local unit normal field). Now, if $z = (x, t)$, one may use (3.1), (3.2) and (8.2) to show

$$D_t f_{*z}\left(\frac{\partial}{\partial t}\right) = \lambda(t)\xi_z, \qquad D_t\xi_z = -\lambda(t) f_*\left(\frac{\partial}{\partial t}\right).$$

Thus, for each $t_1 \in \mathbf{R}$, $\lambda(t)$ is the curvature of $c = f_1(t)$ at $t = t_1$.

If $c\colon \mathbf{L}^1 \to \mathbf{L}^2$ is a unit-speed time-like curve in $\mathbf{L}^2$, then let $e_1 = dc/dt$ be the unit tangent vector, and let e_2 be the unit (space-like) normal to the curve. Then $(e_1, e_2) \in \mathrm{O}(1, 1)$. Since $\langle de_i/dt, e_i\rangle = 0$ $(i = 1, 2)$, (1.3) and (1.5) imply the equations

$$\frac{de_1}{dt} = -k(t)e_2, \qquad \frac{de_2}{dt} = -k(t)e_1. \tag{8.8}$$

In analogy with the case of Euclidean plane curves, k could be called the *curvature* of the curve c.

Now in case (i) of (8.7), at each point $z \in \mathbf{L}^n$, $(f(M_0(z)))^\perp$ is spanned orthonormally by the time-like $f_{*z}(\partial/\partial t) = df_1(t)/dt$ and the space-like unit normal ξ_z. If $z = (x, t)$, one may in this case derive

$$D_t f_*\left(\frac{\partial}{\partial t}\right) = -\lambda(t)\xi_z, \qquad D_t\xi = -\lambda(t) f_*\left(\frac{\partial}{\partial t}\right). \tag{8.9}$$

Since the equations (8.9) fit the form (8.8), $\lambda(t_1)$ can be considered as the curvature of $c = f_1(t)$ at $t = t_1$.

9. The degenerate case ($G = \varnothing$). Choose the origin 0 of $\mathbf{L}^n$ to be in W, and let M_0 be the hyperplane-leaf through 0. M_0 is a degenerate hyperplane, so there is a unique light line contained in M_0, generated by Ω. Now let L be a tangent light vector at 0 such that $\langle L, \Omega\rangle = -1$ and $AL \neq 0$; then $AL = \rho\Omega$. Denote $\operatorname{Span}\{L\}$ by sL ($s \in \mathbf{R}$) and $\operatorname{Span}\{\Omega\}$ by $u\Omega$ ($u \in \mathbf{R}$). Then

$$\mathbf{L}^n = M_0 \oplus sL \tag{9.1}$$

describes $\mathbf{L}^n$ as a direct sum of vector spaces.

Let E be the $(n-2)$-dimensional subspace in M_0 which is the orthogonal complement in $\mathbf{L}^n$ of the Lorentz plane $\operatorname{Span}\{L, \Omega\}$. Then

$$\mathbf{L}^n = E \times \operatorname{Span}\{L, \Omega\}$$

is a decomposition of $\mathbf{L}^n$ both as an orthogonal sum of vector spaces and as a product of pseudo-Riemannian manifolds. We also have a direct sum

$$\mathbf{L}^n = E \oplus u\Omega \oplus sL.$$

Representing a point of $\mathbf{L}^n$ by (p, u, s) ($p \in E$; $u, s \in \mathbf{R}$) provides a global flat chart on $\mathbf{L}^n$.

If $z = (p, u, s) \in W$, then $M_0(z) = \{(q, v, x)\colon q \in E,\ v \in \mathbf{R}\}$ since both sets are hyperplanes parallel to M_0. If $z = (p, u, s) \notin W$, then *define* $M_0(z)$ to be as above, with $T_0(z) = T_z(M_0(z))$. Each $M_0(z)$ contains a unique light line, $\{(p, v, s)\colon v \in \mathbf{R}\}$, spanned by $(\partial/\partial u)_z$. For each $z \neq 0$ in $\mathbf{L}^n$, define

$$E(z) = \{(q, u, s)\colon q \in E\}.$$

Thus, M_0 and its decomposition $M_0 = E \oplus \operatorname{Span}\{\partial/\partial u\}$ are parallel throughout $\mathbf{L}^n$.

If z is any point in $\mathbf{L}^n$, we may say the following: $(\partial/\partial s)_z$ is a tangent light vector at z with $\langle \partial/\partial s, \partial/\partial u\rangle_z = -1$. Also $(A \cdot \partial/\partial s)_z = \rho(z)(\partial/\partial u)_z$ with $\rho(z) \neq 0$ just in case $z \in W$.

PROPOSITION (9.2). *ρ is a constant on each $M(z)$, $z \in \mathbf{L}^n$. Hence, ρ is a function of s.*

PROOF. Let $z \in W$ (otherwise $\rho \equiv 0$ on $M_0(z)$), and let x_t be any geodesic in $M_0(z)$ emanating from z. For each t, write $\rho(t)$ for $\rho(x_t)$, and Ω_t for $(\partial/\partial u)_{x_t}$. Then $(A\partial/\partial s)_{x_t} = \rho(t)\Omega_t$. Applying the derivation of (6.12) in this case gives

$$\frac{d\rho}{dt} - \langle \nabla_s \vec{x}_t, \Omega_t\rangle \rho(t) = 0$$

and, in particular, that $\langle \nabla_s \vec{x}_t, \Omega_t \rangle$ is independent of the extension of $\vec{x}_t$ to a T_0-field near x_t. But $\nabla_s \vec{x}_t \in T_0$ because the M_0's are parallel hyperplanes in $\mathbf{L}^n$. Since Ω_t generates the degenerate line in $T_0(x_t)$, $\langle \nabla_s \vec{x}_t, \Omega_t \rangle = 0$. This implies that $\rho(t) = \rho(z)$ for all t. Q.E.D.

We now turn our attention to the immersion f: $\mathbf{L}^n \to \mathbf{L}^{n+1}$ under consideration. Assume that $f(0) = 0$. Take M_0 as in (9.1) and define $M_1 = sL$. Then $M = \mathbf{L}^n = M_0 \times M_1$ is a direct, though not orthogonal, product of manifolds-with-connection. M has a nondegenerate metric, as does $\mathbf{R}^N = \mathbf{L}^{n+1}$. If $X \in T_0(z) = T_z M_0$ and $Y \in T_z M_1 = \mathrm{Span}\{(\partial/\partial s)_z\}$, then $\alpha(X, Y) = \langle AX, Y \rangle \xi = 0$, where ξ is a local unit normal field. "Moore's Lemma" allows us to write $f(p, u, s) = f_0(p, u) + f_1(s)$ for $(p, u, s) \in E \times u\Omega \times sL = \mathbf{L}^n$.

If x_t is a geodesic line in $M_0(z)$ emanating from z (so that $x_t = z + t\vec{x}_0$) then, as with (8.4), $f(x_t) = z + tf_{*z}(\vec{x}_0)$.

From this follow the next three facts.

PROPOSITION (9.3). *f is an isometry of $M_0(z)$ onto a (degenerate) $(n-1)$-plane in $\mathbf{L}^{n+1}$.*

PROPOSITION (9.4). *f is an isometry of $E(z)$ onto a Euclidean $(n-2)$-plane in the $(n-1)$-plane $f(M_0(z))$.*

PROPOSITION (9.5). *f maps the light line in $M_0(z)$ onto the light line in the $(n-1)$-plane $f(M_0(z))$.*

PROPOSITION (9.6). *Each of $\{f(M_0(z))\}_{z \in \mathbf{L}^n}$ and $\{f(E(z))\}_{z \in \mathbf{L}^n}$ is a parallel family.*

PROOF. If X is a vector field in either T_0 or E, and Y is any vector field on $\mathbf{L}^n$, then

$$D_Y f_* X = f_*(\nabla_Y X) + \langle Y, AX \rangle \xi = f_*(\nabla_Y X) \in f_* T_0 \text{ or } \in f_* E. \quad \text{Q.E.D.}$$

Before considering the general immersion $\mathbf{L}^n \to \mathbf{L}^{n+1}$ (with degenerate relative nullity), we examine the case $n = 2$, where $E = \varnothing$ and $T_0(0) = \mathrm{Span}\{\Omega\}$, so $\mathbf{L}^2 = u\Omega \times sL$. If $z \in \mathbf{L}^2$, then $\mathbf{L}^3$ is spanned by the null frame $(f_{*z}(\partial/\partial s), f_{*z}(\partial/\partial u), \xi_{f(z)})$ where ξ is a local unit (space-like) normal field. It will be evident that we may assume that this frame is proper, without any loss of generality.

Now:

(i) $df_1/ds = f_*(\partial/\partial s)$,

(ii) $D_s f_*(\partial/\partial s) = f_*(\nabla_s \partial/\partial s) + \langle A\partial/\partial s, \partial/\partial s \rangle \xi = -\rho(s)\xi$,

(iii) $D_s f_* \partial/\partial u = \rho(s)\langle \Omega, \Omega \rangle \xi = 0$,

(iv) $D_s \xi = -f_*(A\partial/\partial s) = -\rho(s) f_*(\partial/\partial u)$.

Statements (i)–(iv) say that $(f_1(s), F(s))$, where $F(s)$ is the frame

$(f_*\partial/\partial s, f_*\partial/\partial u, \xi)$, is a framed null curve that in fact is a GNC (cf. equations (3.11)). Since (7.2) and (9.5) imply that the immersion f is given by

$$f(u, s) = f_0(u) + f_1(s) = f_1(s) + uf_*(\partial/\partial u),$$

the immersed image of $\mathbf{L}^2$ under f is the B-scroll of that GNC. We shall call f a "B-scroll immersion" of $\mathbf{L}^2$ into $\mathbf{L}^3$.

(9.7). SECOND CLASSIFICATION THEOREM: SURFACE VERSION. *The isometric immersions $\mathbf{L}^2 \to \mathbf{L}^3$ with degenerate relative nullities are precisely the B-scroll immersions.*

If $n > 2$, then the Lorentz 3-spaces orthogonal to $f(E(z))$ in $\mathbf{L}^{n+1}_{f(z)}$ (for $z \in \mathbf{L}^n$) are parallel in $\mathbf{L}^{n+1}$, by Proposition (9.6). If X is an E-vector field, then $\langle f_*X, f_*\partial/\partial s\rangle = \langle X, \partial/\partial s\rangle = 0$, so $f_1(s)$ lies in the Lorentz 3-space orthogonal to $f(E)$. Also, $\langle f_*X, f_*\partial/\partial u\rangle = 0$. Now write

$$\mathbf{L}^n = \mathbf{E}^{n-2} \times \mathbf{L}^2 = E \times \operatorname{Span}\{\Omega, L\};$$

and write

$$\mathbf{L}^{n+1} = \mathbf{E}^{n-2} \times \mathbf{L}^3 = f(E) \times \operatorname{Span}\{f_*\Omega, f_*L, \xi_0\}$$

where ξ_0 is a unit normal at $0 \in \mathbf{L}^{n+1}$. Then f factors

$$f|_E \times f|_{\mathbf{L}^2}\colon \mathbf{E}^{n-2} \times \mathbf{L}^2 \to \mathbf{E}^{n-2} \times \mathbf{L}^3.$$

By Proposition (9.4), we may assume that, up to a proper Lorentz motion of $\mathbf{L}^{n+1}$, $f|_E$ is the identity map of $\mathbf{E}^{n-2}$ onto itself. Proposition (9.7) applies to $f|_{\mathbf{L}^2}$.

(9.8). SECOND CLASSIFICATION THEOREM: MAIN VERSION. *Up to a proper motion of $\mathbf{L}^{n+1}$, the isometric immersions: $\mathbf{L}^n \to \mathbf{L}^{n+1}$ with degenerate relative nullities have the form*

$$\mathbf{E}^{n-2} \times \mathbf{L}^2 \xrightarrow{\mathrm{id}\times g} \mathbf{E}^{n-2} \times \mathbf{L}^3$$

where the factors in each product are orthogonal, and the map g: $\mathbf{L}^2 \to \mathbf{L}^3$ is a B-scroll immersion.

We remark that the hypersurfaces described by (8.7) and the Hartman-Nirenberg theorem are "built" over plane curves, whereas those described by (9.7) and (9.8) are "built" over certain curves in a 3-dimensional space. That space curves rather than plane curves are required there is the import of the following observation, whose straightforward proof is omitted.

PROPOSITION (9.9). *Suppose $\mathbf{L}^2 = \operatorname{Span}\{\xi, Y\}$ with ξ a light vector and $\langle \xi, Y\rangle = k$, a constant. Suppose η is a light vector in $\mathbf{L}^3$ such that*

$$f(u, v) = f(u\xi + vY) = x(v) + u\eta$$

is an isometric immersion: $\mathbf{L}^2 \to \mathbf{L}^3$. *If* $x(v)$ *is a plane curve, then* $x(v)$ *is a line, and hence the immersion is totally geodesic*.

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